Numerical investigation of effective stability and chaos in the J_2–Solar radiation pressure problem

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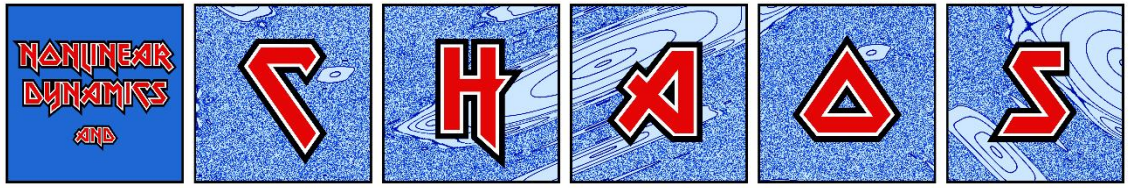

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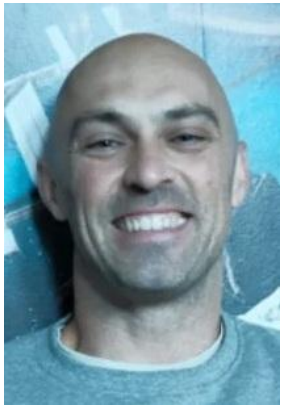

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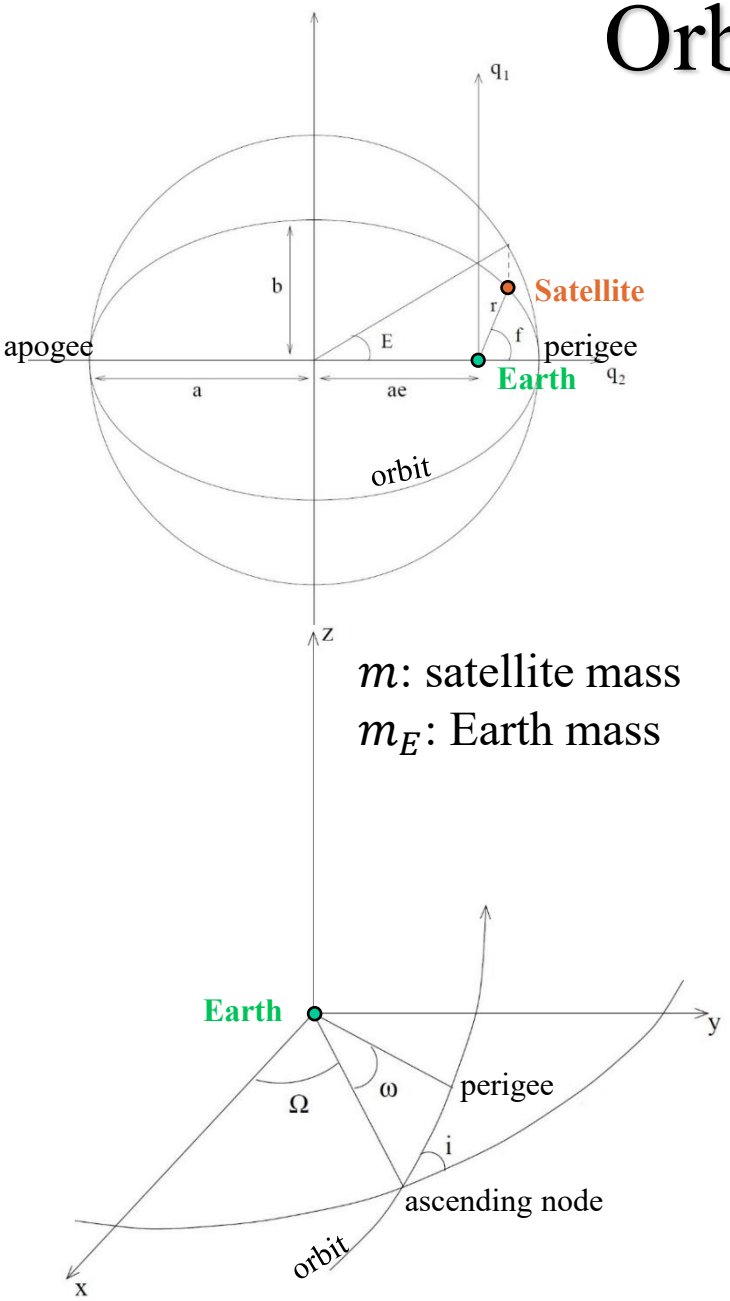
Barbis, Daquin, Alessi & S.: 'Regularity and reentry basins of low Earth orbits in the J_2-solar radiation pressure problem', arXiv:nlin.CD/2605.24575 (2026)

Outline

- Motivation and problem statement:
  In the context of the space debris problem, we extend previous studies by investigating how the coupled effect of solar radiation pressure (SRP) and Earth's oblateness (J_2) can drive chaotic eccentricity growth and enable passive debris reentry.
- Hamiltonian system:
  Perturbations of the two-body problem due to SRP and Earth's oblateness effect.
- Chaos detection:
  The Smaller ALignment Index (SALI).
- Parametric cartography and numerical investigations:
  SALI and orbital lifetime.
- Re-entry basins:
  Smooth and fractal boundaries.
- Summary.

The Hamiltonian model

Orbital elements



m: satellite mass

m_E: Earth mass

Figures adopted from: Morbidelli, 'Modern celestial mechanics: aspects of solar system dynamics' (2011).

Keplerian motion

f: true anomaly

E: eccentric anomaly

$q_1 = a(\cos E - e)$,

$q_2 = a\sqrt{1-e^2}\sin E$,

$r = (1 - e\cos E)$, $\quad \cos f = \frac{\cos E - e}{1 - e\cos E}$

M: mean anomaly

$\mathcal{G}$: gravitational constant

t: time

t_0: time of passage at perigee

$M = \sqrt{\mathcal{G}(m_E + m)}a^{-3/2}(t - t_0)$

Orbital elements

a: semi-major axis

e: eccentricity

i: inclination

ω: argument of perigee

Ω: longitude of ascending node

M: mean anomaly

Hamiltonian model

Following Alessi et al., MNRAS, 473, 2407 (2018) and Gkolias et al., Celest. Mech. Dyn. Astron., 132, 55 (2020) the motion of a satellite (mass m) around Earth (mass m_E, radius r_E) is described by a perturbed two-body secular Hamiltonian where the Earth's oblateness (J_2) and the solar radiation pressure (SRP) are taken into account.

- Cannonball model: the Sun's radiation is normal to the surface of the satellite.
- Effects of Earth's albedo and shadowing are neglected.
- Perturbations averaged in closed form over the mean anomaly M.

$$\mathcal{H} = \frac{\mu r_E^2 J_2 (1 - 3\cos^2 i)}{4a^3(1-e^2)^{3/2}} - \frac{3}{2} a e c_R P \frac{A}{m} \sum_{k=1}^{6} \mathcal{T}_k(i) \cos \psi_k(\omega, \Omega, \lambda_\odot)$$

$J_2 = 0.0010826$ (oblateness parameter)
$\mu = \mathcal{G} m_E$ (Earth's gravitational parameter), we consider $m \ll m_E$.
$c_R = 1$ (reflectivity coefficient)
$P = 4.56 \times 10^{-6}$ (SRP constant at 1 AU)
A/m: area-to-mass ratio
$\varepsilon = 23.4°$(ecliptic's obliquity)
$\lambda_\odot = \lambda_{\odot,0} + n_\odot t$ (Sun's ecliptic longitude)
$\lambda_{\odot,0}$ determined from astronomical ephemerides
$n_\odot = 2\pi/(86400 \times 365.25)\ \mathrm{s}^{-1}$ (Sun's mean motion)

$\mathcal{T}_k(i)$, $k = 1, \ldots, 6$: coefficients of the 6 SRP-induced harmonics

k	$\mathcal{T}_k(i)$	$\psi_k(\omega, \Omega, \lambda_\odot)$
1	$\frac{1}{4}(\cos\varepsilon + 1)(\cos i + 1)$	$\omega + \Omega - \lambda_\odot$
2	$-\frac{1}{4}(\cos\varepsilon + 1)(\cos i - 1)$	$-\omega + \Omega - \lambda_\odot$
3	$\frac{1}{2}\sin i \sin\varepsilon$	$\omega - \lambda_\odot$
4	$-\frac{1}{2}\sin i \sin\varepsilon$	$\omega + \lambda_\odot$
5	$-\frac{1}{4}(\cos\varepsilon - 1)(\cos i + 1)$	$\omega + \Omega + \lambda_\odot$
6	$\frac{1}{4}(\cos\varepsilon - 1)(\cos i - 1)$	$-\omega + \Omega + \lambda_\odot$

Hamiltonian model

We introduce canonical Delaunay variables (L, G, H, ℓ, g, h)

$$L = \sqrt{\mu a}, \qquad l = M,$$

$$G = L\sqrt{1 - e^2}, \qquad g = \omega,$$

$$H = G\cos i\,, \qquad h = \Omega.$$

We eliminate the system's explicit time dependency due to $\lambda_\odot = n_\odot t + \lambda_{\odot,0}$, by introducing a dummy action variable J conjugate to a variable s such that $s(t) = n_\odot t + \lambda_{\odot,0}$.
Thus, we obtain an autonomous system with 3 degrees of freedom (Dof) in generalized positions (g, h, s) and momenta (G, H, J).

$$\mathcal{H} = \frac{r_E^2 J_2 \mu^4 (G^2 - 3H^2)}{4G^5 L^3} + n_\odot J -$$

$$- \frac{3c_R P L^2}{2\mu} \frac{A}{m} \sqrt{1 - \frac{G^2}{L^2}} \sum_{k=1}^{6} \mathcal{T}_k\big(i(L, G, H)\big) \cos \psi_k(g, h, s)$$

Note that l is a cyclic coordinate, therefore L and a are constants.

The Smaller ALignment Index (SALI) method of chaos detection

Maximum Lyapunov Exponent (MLE)

Chaos: sensitive dependence on initial conditions.

Roughly speaking, the MLE of a given orbit characterizes the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the *2N*-dimensional phase space with initial condition *x(0)* and an initial deviation vector *v(0)* (small perturbation) from it.
Then the mean exponential rate of divergence is:

$$MLE = \lambda_1 = \lim_{t\to\infty} \Lambda(t) = \lim_{t\to\infty} \frac{1}{t} ln \frac{\|v(t)\|}{\|v(0)\|}$$

$\lambda_1 = 0 \rightarrow$ Regular motion ($\Lambda \propto t^{-1}$)

$\lambda_1 > 0 \rightarrow$ Chaotic motion

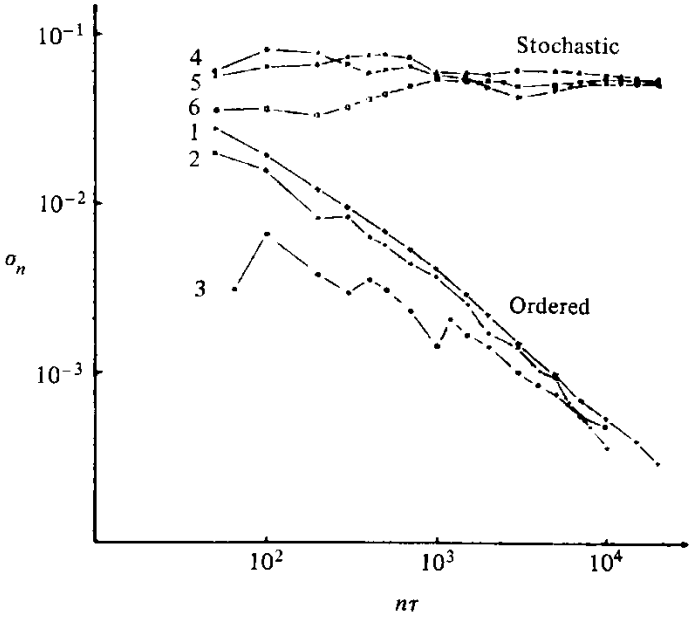


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

Definition of the SALI

We follow the evolution in time of two different initial deviation vectors $v_1(0)$, $v_2(0)$ and define SALI [S., J. Phys. A. 34, 10029 (2001) – S. & Manos, Lect. Notes Phys., 915, 129 (2016)] as:

$$SALI(t) = min\{\|\hat{v}_1(t) + \hat{v}_2(t)\|, \|\hat{v}_1(t) - \hat{v}_2(t)\|\}$$

where

$$\hat{v}_1(t) = \frac{v_1(t)}{\|v_1(t)\|}$$

When the two vectors become collinear

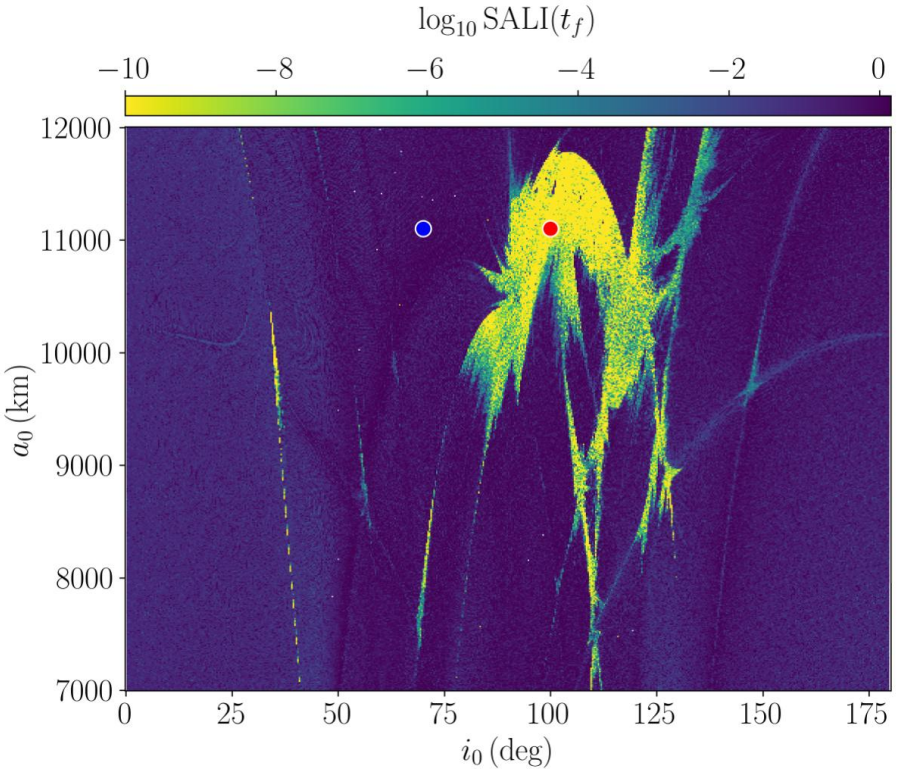
$$SALI(t) \rightarrow 0$$

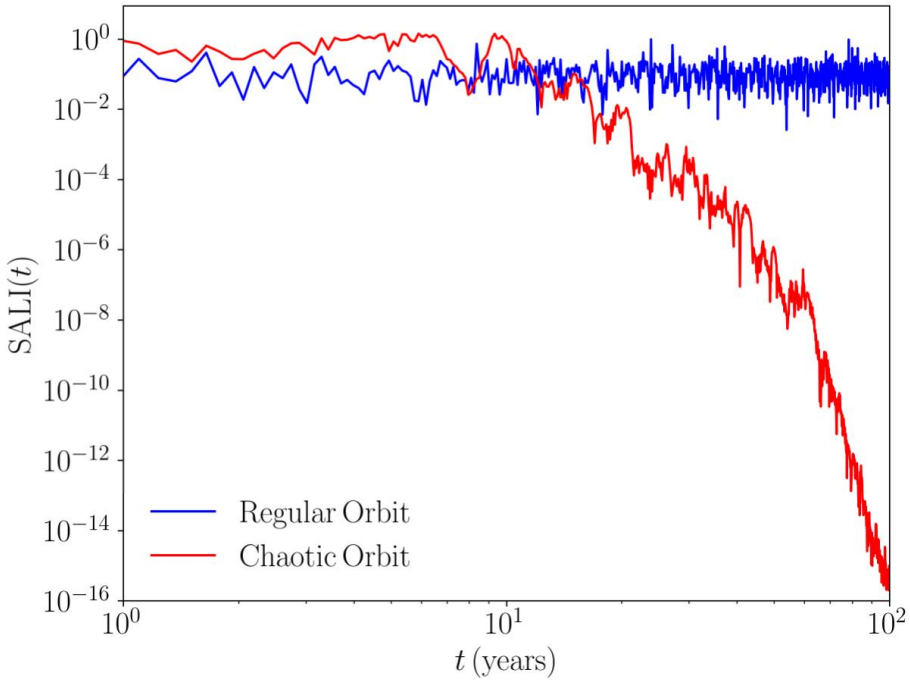
Behavior of the SALI

Hamiltonian flows and multidimensional maps

SALI→0 exponential decay for chaotic orbits

SALI→constant ≠ 0 for regular orbits

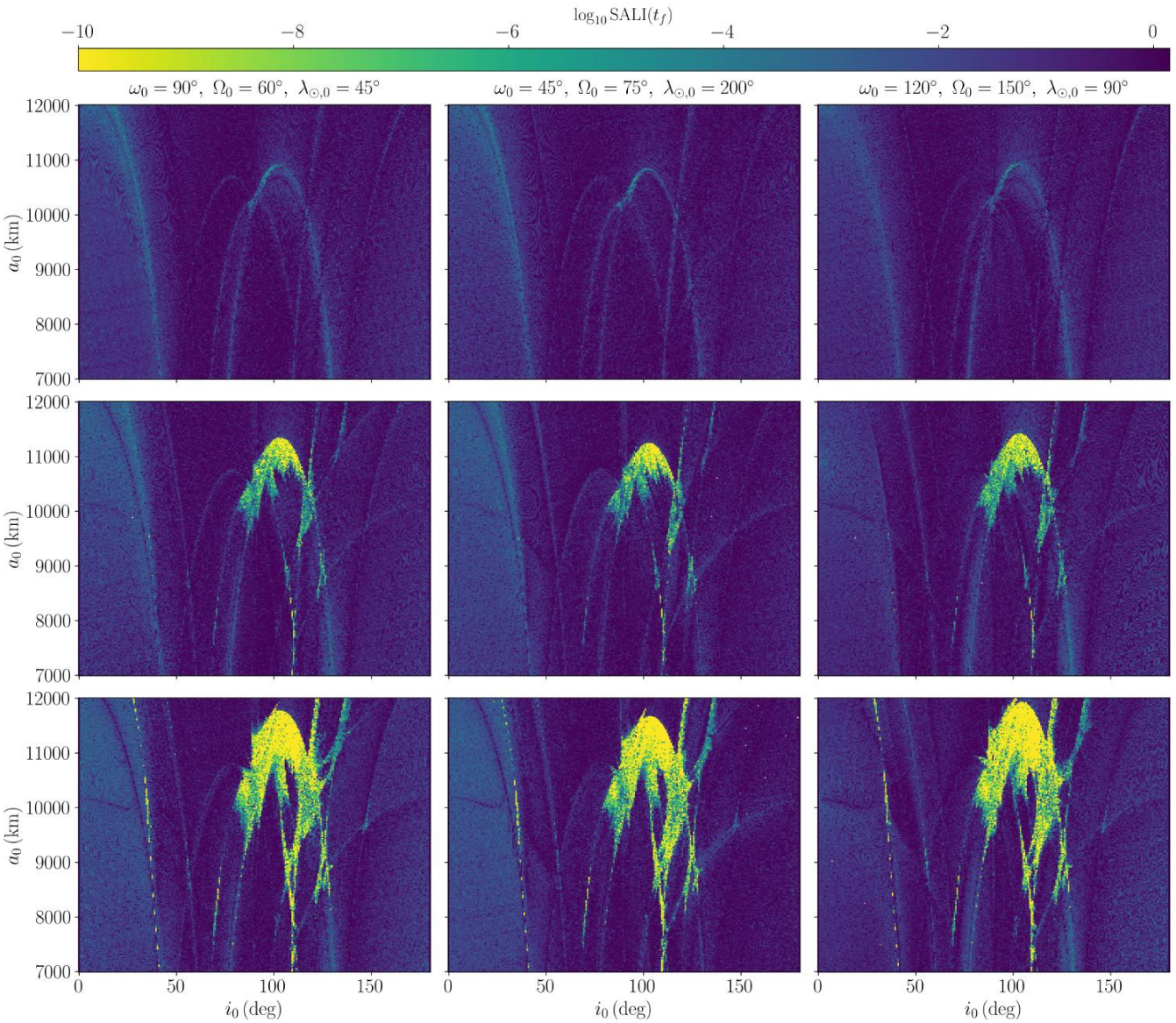




We consider initial conditions (ICs) with $e_0 = 0.05$ and $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0°$, for $A/m = 10\ \mathrm{m^2 Kg^{-1}}$. We register the SALI at $t_f = 100$ years.

Numerical investigations

Effect of ω_0, Ω_0 and $\lambda_{\odot,0}$



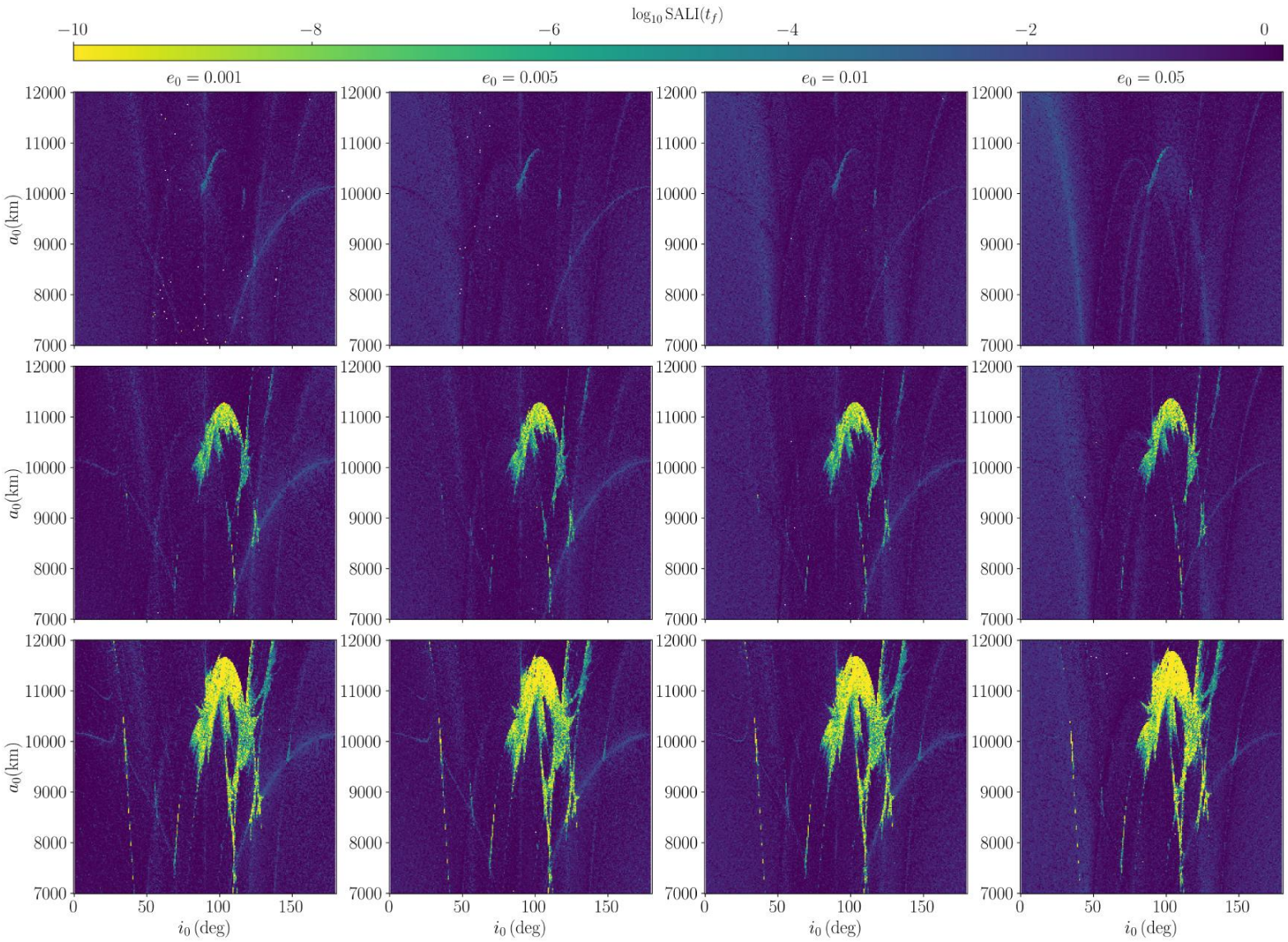
$A/m = 1\ \mathrm{m^2Kg^{-1}}$

$A/m = 5\ \mathrm{m^2Kg^{-1}}$

$A/m = 10\ \mathrm{m^2Kg^{-1}}$

ICs with $e = 0.05$. SALI at $t_f = 100$ years.

Effect of eccentricity



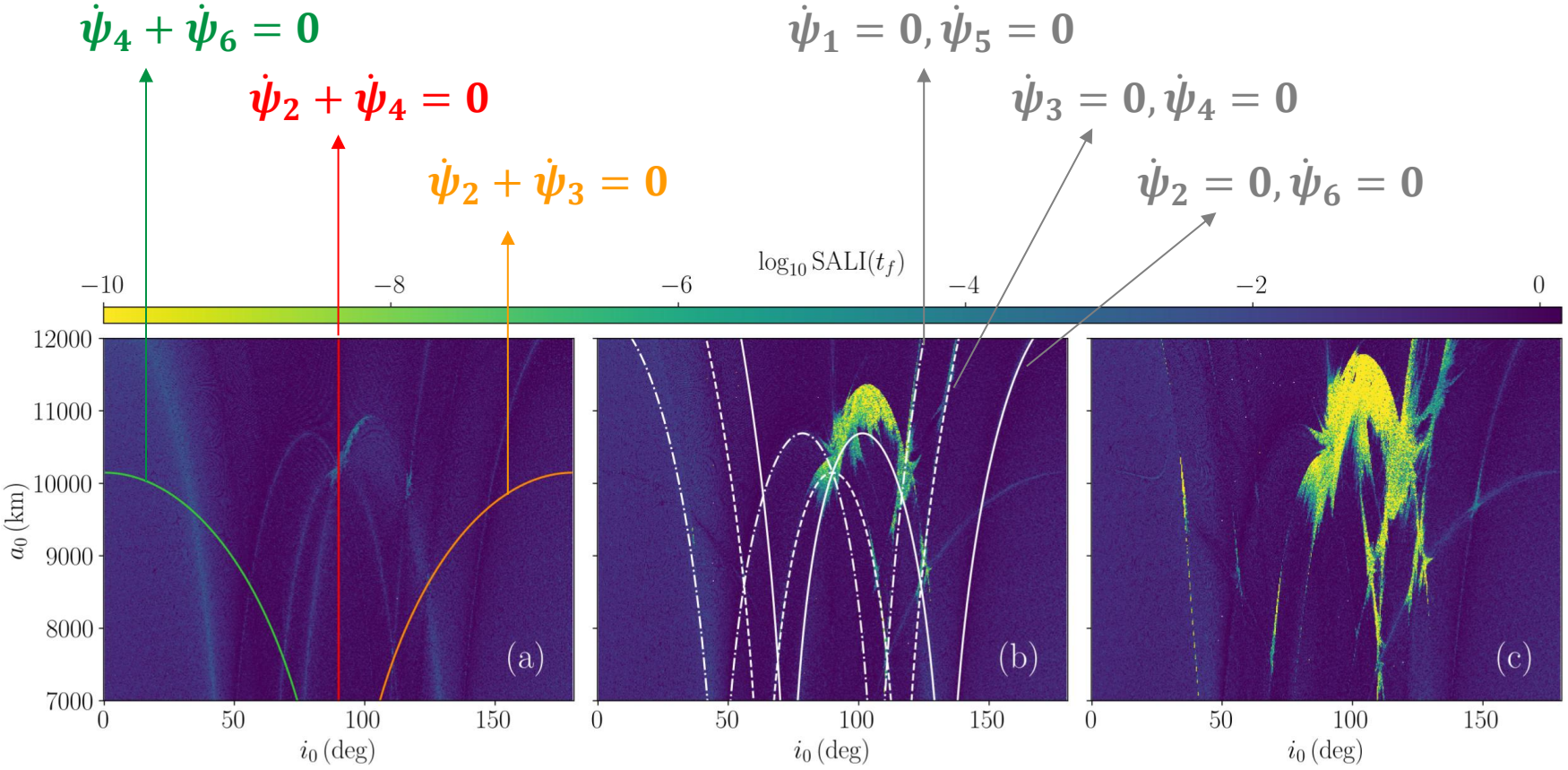
$A/m = 1\ \mathrm{m^2Kg^{-1}}$

$A/m = 5\ \mathrm{m^2Kg^{-1}}$

$A/m = 10\ \mathrm{m^2Kg^{-1}}$

$\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0°$ and $t_f = 100$ years.

Effect of area-to-mass ratio



$A/m = 1\ \mathrm{m^2Kg^{-1}}$ $A/m = 5\ \mathrm{m^2Kg^{-1}}$ $A/m = 10\ \mathrm{m^2Kg^{-1}}$

ICs with $e_0 = 0.05$, $\omega_0 = \Omega_0 = \lambda_{\odot,0} = 0°$. SALI at $t_f = 100$ years.

Resonances are computed by considering only the Earth's oblateness perturbation.

The extent of chaotic motion increases with increasing A/m values.

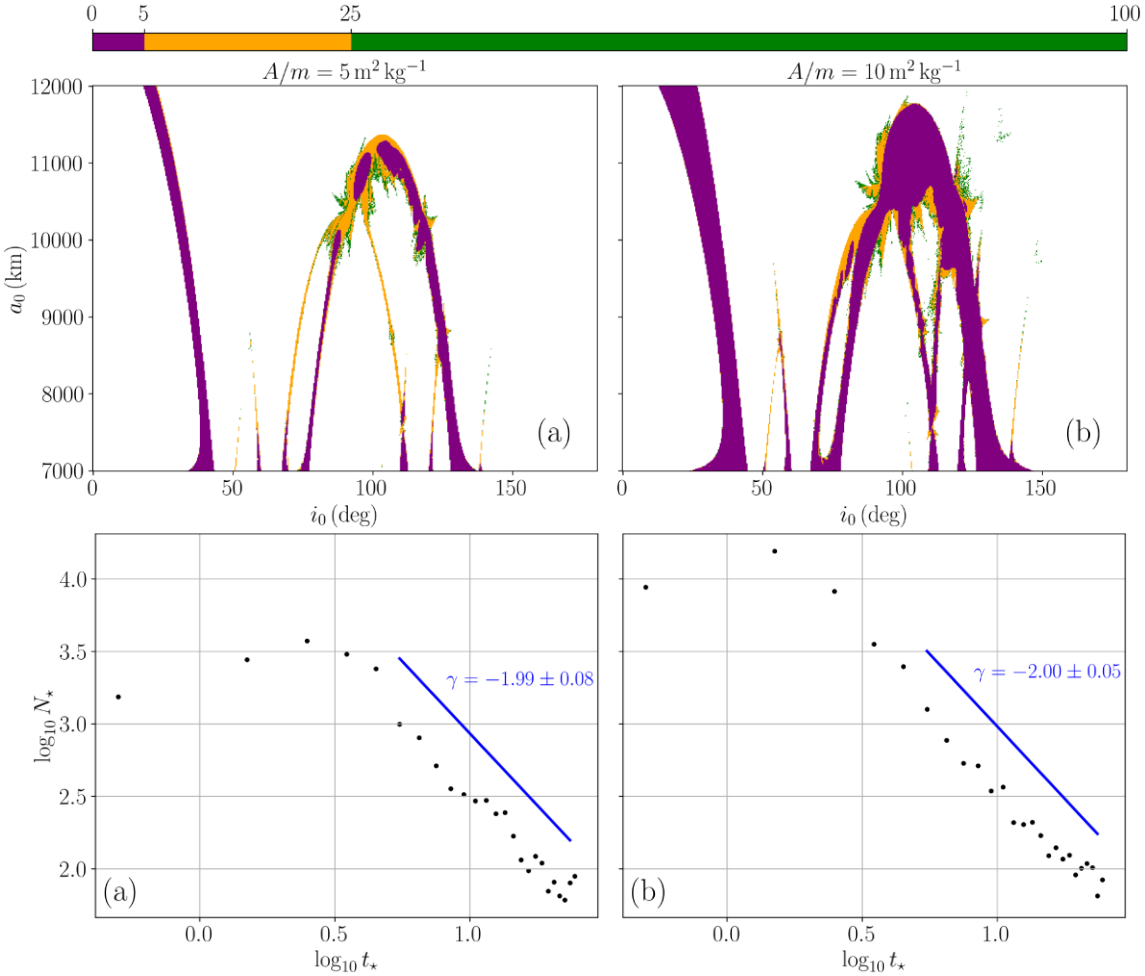
Orbital lifetime and re-entry trajectories

Orbital lifetime t_*: the time an orbit with initial semi-major axis a_0 needs to reach the critical eccentricity $e_* = 1 - \frac{r_E+\delta}{a_0}$, [(perigee altitude) $\geq r_E + \delta$] ($\delta = 120$Km and $r_E = 6378.1363$Km). At that time the orbit is considered a re-entry trajectory.

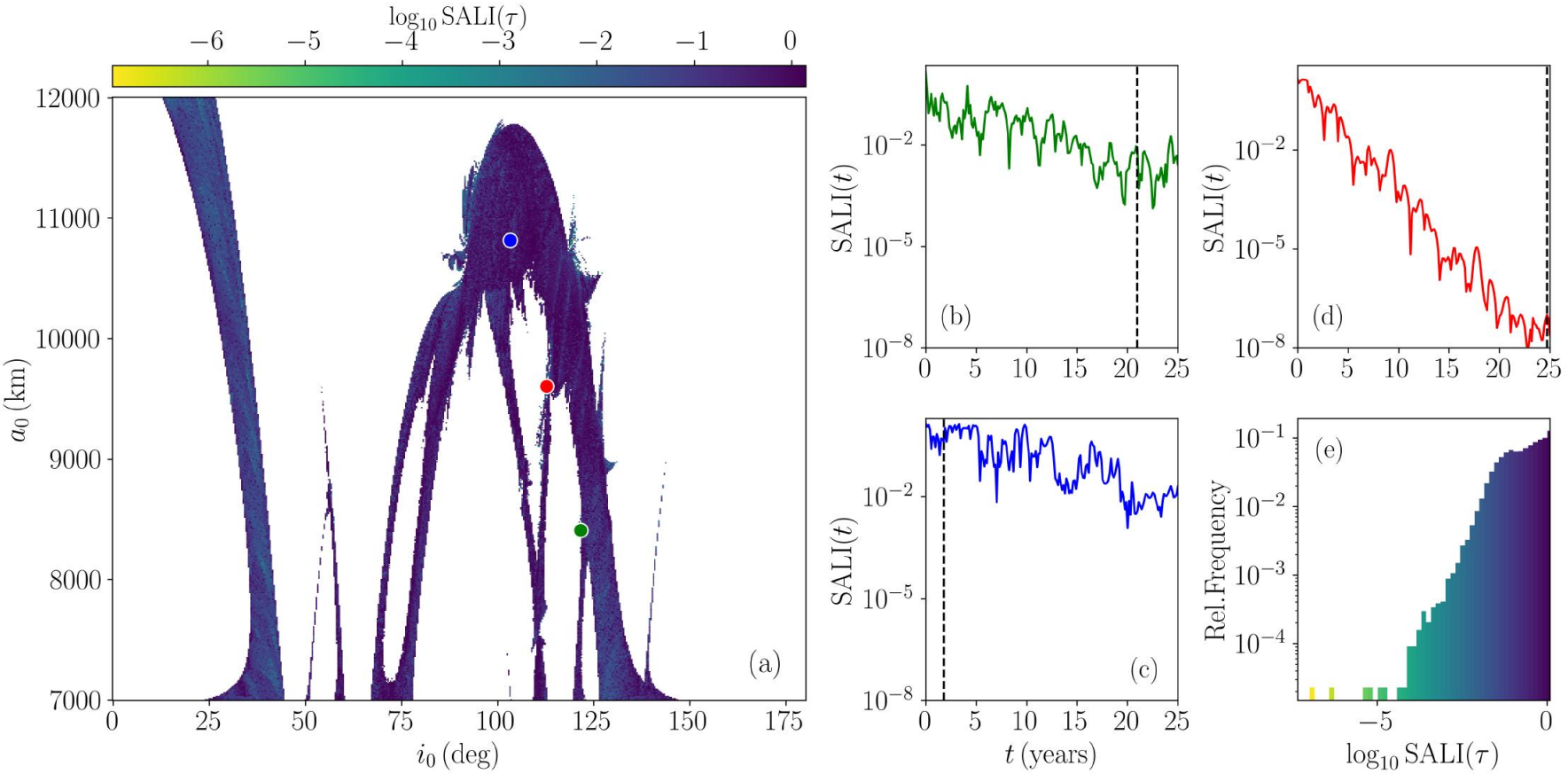
The majority of re-entry orbits are associated with re-entry times shorter than 5 years.

$$N_* \sim t_*^{-2}$$

N_*: the number of ICs (out of about 260,000 considered orbits) leading to re-entry for times less than 25 years.



Chaoticity and orbital lifetimes

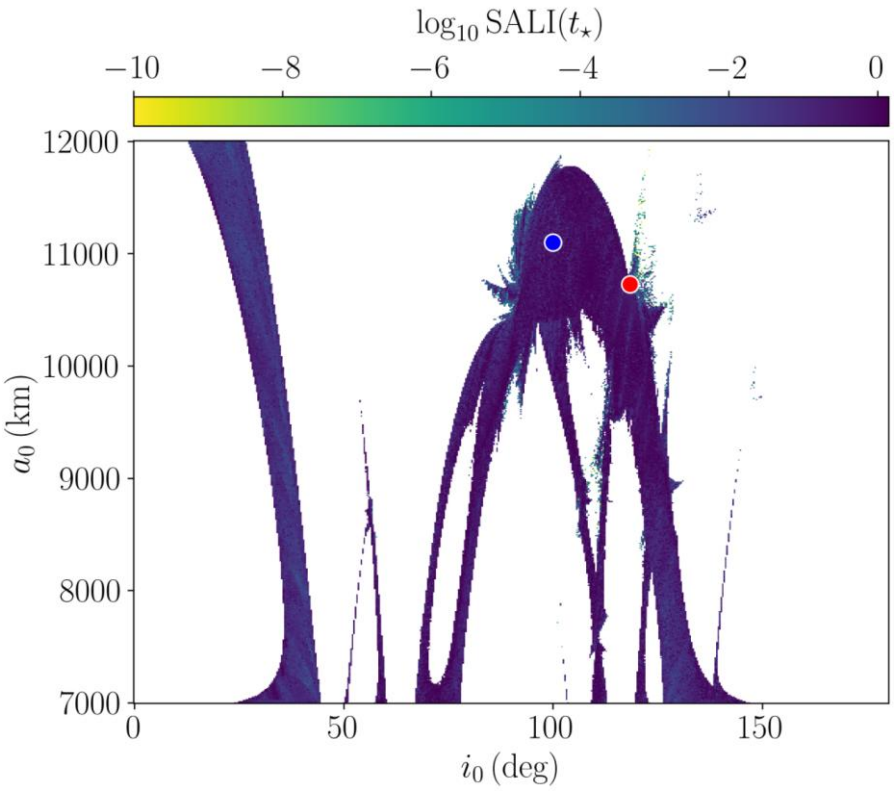


SALI is computed at the effective time τ, which is the orbital lifetime t_* of orbits with $t_* \leq 25$ years.

Almost all re-entry orbits exhibit regular behavior until the time of re-entry.

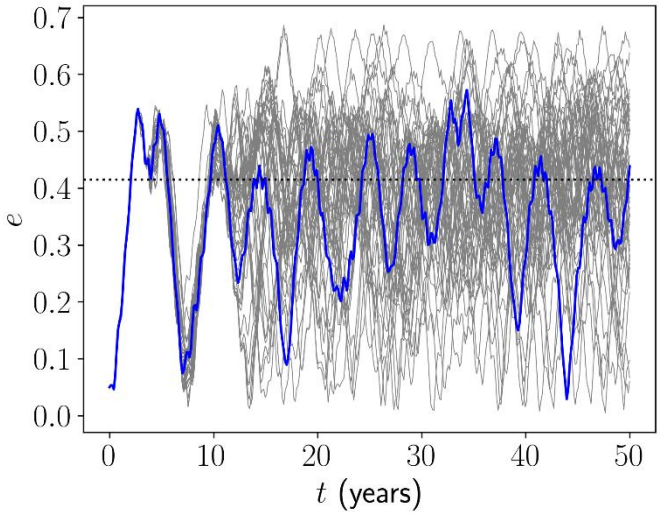
Chaoticity and orbital lifetimes

Orbits with $t_* \leq 100$ years.

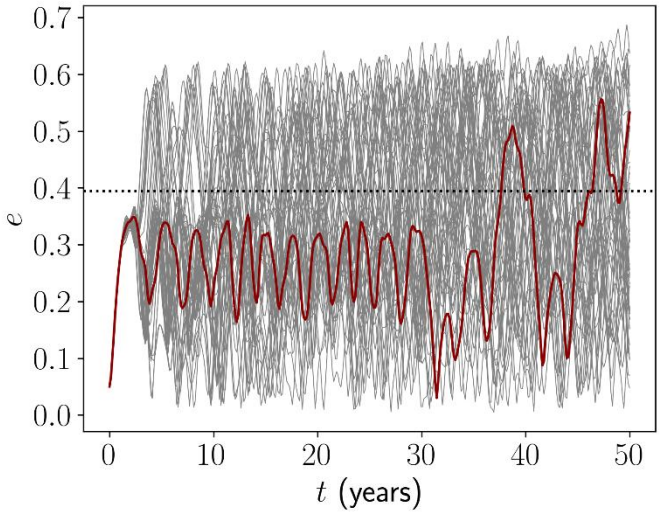


$(a, e, i, \omega, \Omega) = (11100\text{Km}, 0.05, 100°, 0°, 0°)$

$(a, e, i, \omega, \Omega) = (10727.98\text{Km}, 0.05, 118.36°, 0°, 0°)$



e_*

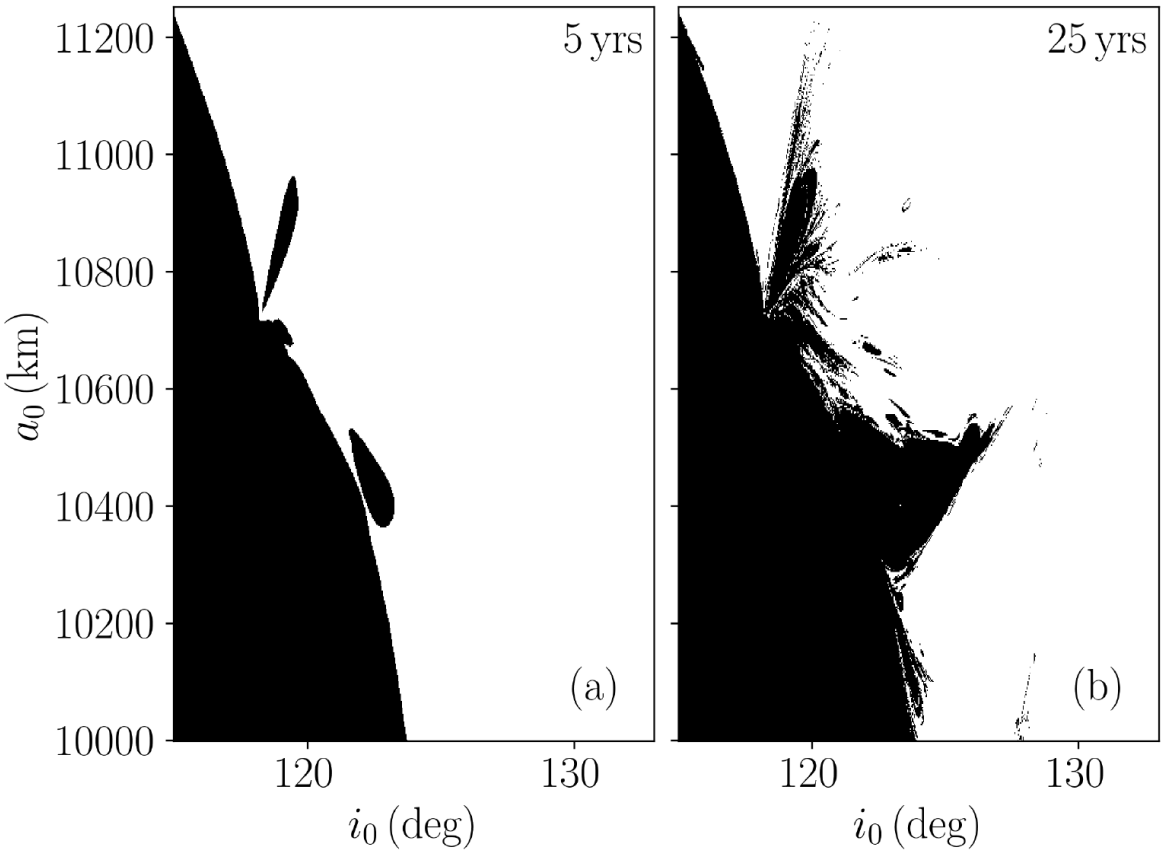


e_*

We consider 50 randomly generated ICs (grey curves) within the neighborhood of the nominal orbit.

Re-entry basins

We set $A/m = 10\ \mathrm{m^2 Kg^{-1}}$ and consider sets of re-entry points following the 5-year and 25-year mitigation scenario.



Basin boundaries look fractal for the less-stringent mitigation scenario.

Uncertain points and uncertainty exponent

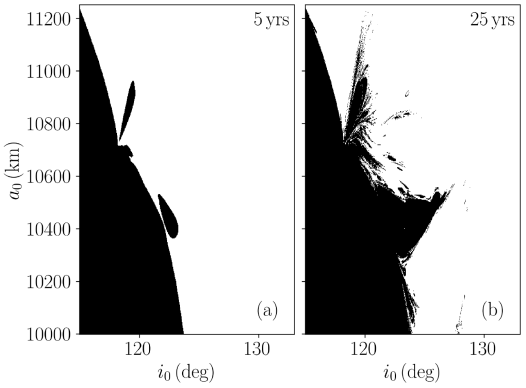
Following Grebogi et. al., Phys. Lett. A, 99, 415 (1983) an IC $\boldsymbol{x_0}$ is ϵ-uncertain if two nearby ICs $\boldsymbol{x_0} \pm \epsilon$ (we perturb only the inclination i by ϵ) do not show the same behavior with respect to the considered mitigation scenario.

$N_u(\epsilon)$: number of ϵ-uncertain ICs, N: number of considered ICs
$f(\epsilon)$: uncertainty fraction, α: uncertainty exponent

$$f(\epsilon) = \frac{N_u(\epsilon)}{N} \propto \epsilon^{\alpha}$$

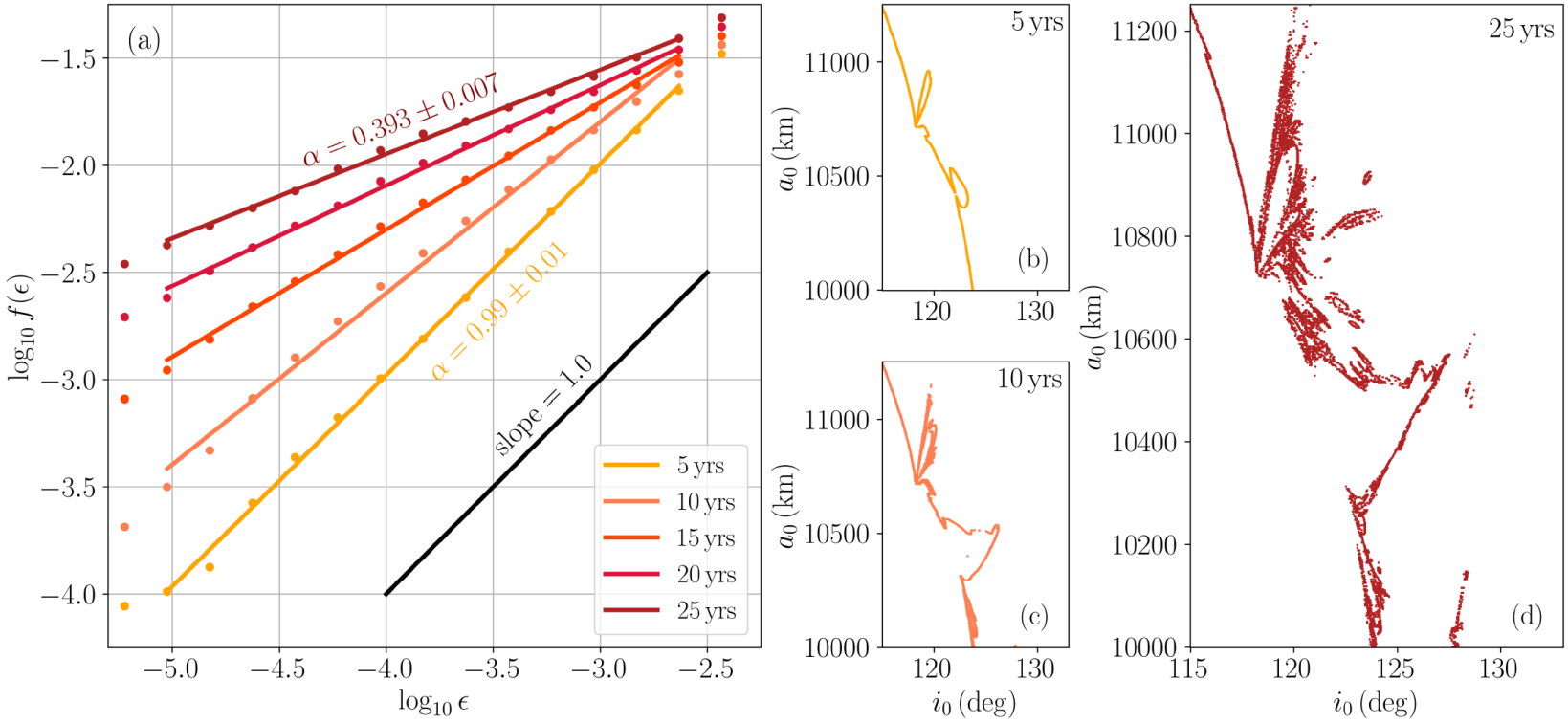
$\alpha = 1$: Smooth boundaries
$0 < \alpha < 1$: Fractal boundaries.
A substantial improvement in the IC uncertainty ϵ, yields only a relative small decrease in the uncertainty of the final state as measured by f.
Example: For $\alpha = 0.5$ reducing the uncertainty in the ICs by a factor of 100 decreases the fraction of uncertain points by only a factor of 10.

Uncertain points and uncertainty exponent

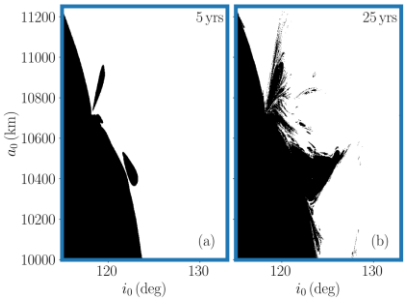


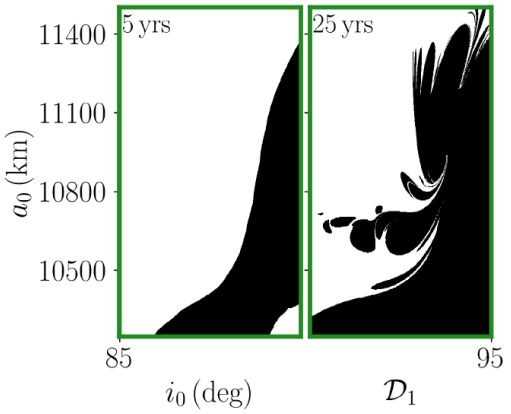
The set of uncertain points develops a significantly more intricate structure for less-stringent mitigation scenarios.

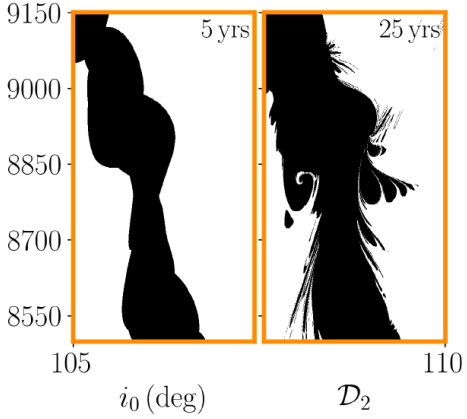
Uncertain points for $\epsilon \approx 0.034°$

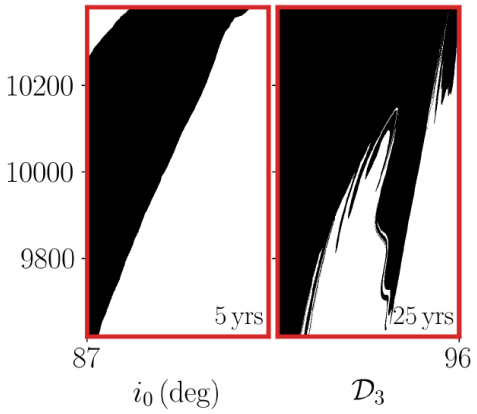


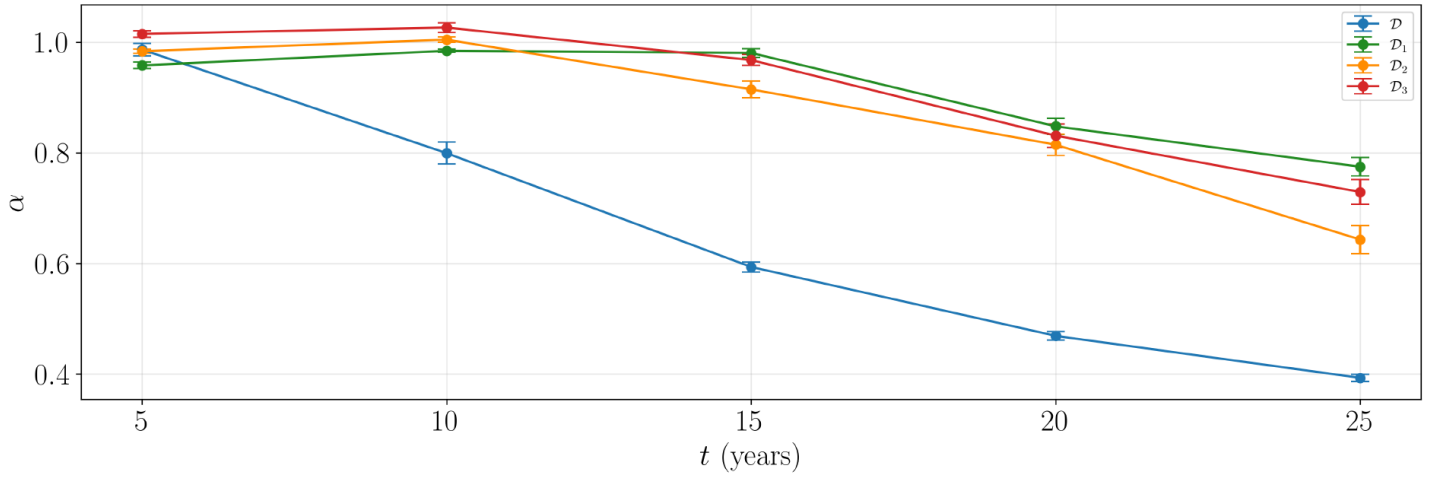
Uncertain points and uncertainty exponent











The value of the uncertainty exponent α decreases (fractality increases) for less-stringent mitigation scenarios.

Summary

- We conducted a numerical investigation of Earth satellite orbital dynamics under perturbations to the two-body Kepler problem arising from solar radiation pressure and the Earth's oblateness.
- The Smaller ALignment Index method (SALI) allows the fast and clear identification of chaotic regions.
- The extent of chaos increases with increasing A/m values.
- The number N_* of re-entry trajectories follows a power law decay with respect to the orbital lifetime t_*: $N_* \sim \frac{1}{t_*^2}$.
- Almost all re-entry trajectories with $t_* \leq 25$ years exhibit regular behavior until the time of re-entry.
- Boundaries of re-entry basins become increasingly fractal (lower uncertainty exponent) as the mitigation timescale increases from 5 to 25 years.
- From an operational perspective targeting the 5-year re-entry basin is easier than targeting the 25-year basin.